

The Potential of Effective Field Theory in NN Scattering

S.R. Beane, T.D. Cohen and D.R. Phillips¹

Department of Physics, University of Maryland, College Park, MD, 20742-4111

Abstract

We study an effective field theory of interacting nucleons at distances much greater than the pion's Compton wavelength. In this regime the NN potential is conjectured to be the sum of a delta function and its derivatives. The question we address is whether this sum can be consistently truncated at a given order in the derivative expansion, and systematically improved by going to higher orders. Regularizing the Lippmann-Schwinger equation using a cutoff we find that the cutoff can be taken to infinity only if the effective range is negative. A positive effective range—which occurs in nature—requires that the cutoff be kept finite and below the scale of the physics which has been integrated out, i.e. $O(m_\pi)$. Comparison of cutoff schemes and dimensional regularization reveals that the physical scattering amplitude is sensitive to the choice of regulator. Moreover, we show that the presence of some regulator scale, a feature absent in dimensional regularization, is essential if the effective field theory of NN scattering is to be useful. We also show that one can define a procedure where finite cutoff dependence in the scattering amplitude is removed order by order in the effective potential. However, the characteristic momentum in the problem is given by the cutoff, and not by the external momentum. It follows that in the presence of a finite cutoff there is no small parameter in the effective potential, and consequently no systematic truncation of the derivative expansion can be made. We conclude that there is no effective field theory of NN scattering with nucleons alone.

¹Email: sbeane,cohen,phillips@quark.umd.edu

1 Introduction

There exist many nucleon-nucleon potentials which reproduce phase shifts and nuclear properties with remarkable accuracy [1, 2, 3, 4, 5, 6, 7, 8]. Three fundamental features are shared by these potential models: (i) pions dominate at long distances, (ii) there is some source of intermediate-range attraction, and (iii) there is some source of short-distance repulsion. However, in general, distinct physical mechanisms in these models account for the same feature of the nuclear force. Agreement with experiment is maintained in spite of these differences because of the large number of fit parameters. It would be a considerable advance if a systematic approach to the nucleon-nucleon interaction could be developed based solely on symmetries and general physical principles.

Systematic approaches to the scattering of strongly interacting particles, such as chiral perturbation theory, are based on the ideas of effective field theory (EFT). Effective field theory says that for probes of a system at momentum $k \ll M$, details of the dynamics at scale M are unimportant. What is important at low energies is the physics that can be captured in operators of increasing dimensionality which take the form of a power-series in the quantity k/M [9, 10]. It is important to realize that even if EFT ideas can be applied to the NN system they will probably not lead to any startlingly new predictions for NN scattering. Indeed, it is entirely possible that the resulting fits to phase shifts will not be as good as those produced by conventional NN potentials with the same number of parameters. However, the real motivation for constructing an EFT is to relate one process to another. For instance, one would like to relate NN scattering systematically to scattering processes with more nucleons, such as NNN and $NNNN$ scattering, and to say something predictive about processes involving pionic and photonic probes of few-nucleon systems.

One reason to hope that this can be achieved in nuclear physics is provided by the pattern of chiral symmetry breaking in QCD. The fact that chiral symmetry is spontaneously broken implies that the pion is light and interacts weakly at low energies. Of course the lightness of the pion in itself guarantees that it should play a fundamental role in nuclear physics. The weakness of pion interactions at low energies allows pion interactions with nucleons to be systematized using chiral perturbation theory. This procedure has proved remarkably successful in describing the interactions of pions with a single nucleon [11]. These ideas have been extended to processes involving more than one nucleon, leading to arguments concerning the relative importance of various pion-exchange mechanisms in the nuclear force [12, 13, 14]. However, these EFT arguments assume the existence of a systematic power-counting scheme for multi-nucleon processes. In our view there is no convincing argument in the literature that such a power counting exists. The purpose of this paper is to investigate this issue in the simplest possible context.

Naive transposition of EFT ideas to nuclear physics immediately suggests a puzzle. In nuclear physics there are bound states whose energy is unnaturally small on the scale of hadronic physics. In order to generate such bound states within a “natural” theory it is clear that one must sum a series to all orders. Therefore, Weinberg proposed [12, 13] implementing the EFT program in nuclear physics by applying the power-counting arguments of chiral perturbation theory to an n -nucleon effective potential rather than directly to the S-matrix. Only n -nucleon irreducible graphs should be included in the n -nucleon effective potential.

The potential obtained in this way is then to be inserted into a Lippmann-Schwinger or Schrödinger equation and iterated to all orders. There will of course be unknown coefficients in the effective potential, but these can be fit to experimental data as in ordinary chiral perturbation theory [15, 16, 17, 18].

Thus, an EFT treatment of the NN interaction differs in a fundamental way from conventional EFT applications like $\pi\pi$ scattering in chiral perturbation theory. In both cases operators are ordered in an effective Lagrangian in the same way. However, in $\pi\pi$ scattering the operator expansion in the effective Lagrangian maps to a power series in k/M in the scattering amplitude. It is straightforward to see that EFT treatments where there is a direct mapping from the Lagrangian to the S-matrix are systematic [19]. On the other hand, when the mapping is from the Lagrangian to an effective potential which is subsequently iterated to all orders, many issues arise which lead one to question the existence of a systematic power counting in the potential.

In order to raise some of these issues, consider NN scattering in the 1S_0 channel at momentum scales $k \ll m_\pi$. The EFT at these scales involves only nucleons since the pion is heavy and may therefore be “integrated out”. The effective Lagrangian thus consists of contact operators of increasing dimensionality constrained by spin and isospin. Throughout the paper this is the EFT which we consider. We do not intend that this EFT should provide a quantitative description of the NN phase shifts. Instead, we study it because scattering amplitudes can be calculated analytically. It therefore allows us to elucidate issues of principle in EFT for NN scattering.

One might naively expect to be able to calculate the NN scattering amplitude directly from the effective Lagrangian as a power series in k/m_π , where m_π is the heavy scale in this problem. It is instructive to show in some detail why this fails and one is led to consider an expansion in the potential. Consider an expansion of the amplitude in the S -wave channels:

$$T^{\text{on}}(k, k) = c_0 + c_2 \left(\frac{k}{m_\pi} \right)^2 + c_4 \left(\frac{k}{m_\pi} \right)^4 + \dots, \quad (1)$$

where $k^2 = ME$, and, with the prevailing prejudice of EFT, we anticipate that the dimensionless coefficients c_0, c_2 , etc. will be natural; i.e. of order unity. We know that in the 1S_0 and $^3S_1 - ^3D_1$ channels there are, respectively, a quasi-bound state and a bound state at low energies. The power series expansion (1) with natural coefficients can only be correct if these bound states are at energies $k^2 \sim m_\pi^2$. However, in these channels the bound states occur at unnaturally low energies, i.e. at energies $k^2 \ll m_\pi^2$. Therefore, the coefficients in the expansion must be unnatural—they are fixed by the pole positions of the low-lying bound states rather than by the scale of the physics that has been integrated out. This limits the usefulness of an expansion in the amplitude to an extremely restricted domain of validity. On the other hand, if one makes the following expansion of the potential in S -wave channels:

$$V(p', p) = c_0 + c_2 \frac{p^2 + p'^2}{m_\pi^2} + c_4 \frac{p^4 + p'^4}{m_\pi^4} + c'_4 \frac{p^2 p'^2}{m_\pi^4} + \dots, \quad (2)$$

and iterates it via the Lippmann-Schwinger equation (see Fig. 1)

$$T(p', p; E) = V(p', p) + M \int \frac{d^3 q}{(2\pi)^3} V(p', q) \frac{1}{EM - q^2 + i\epsilon} T(q, p; E), \quad (3)$$

one may hope to generate (quasi-)bound states at the appropriate energies while maintaining natural coefficients in the potential. At face value this procedure appears promising. The expansion (2) may be truncated at some finite order in the quantities p/m_π and p'/m_π and, provided $p, p' \ll m_\pi$ the neglected terms will be small. However, in making the expansion in the potential of Eq. (2) and then iterating, a number of issues arise which are absent in standard EFT treatments, and which render this procedure suspect. In what follows we will identify and clarify some of these issues.

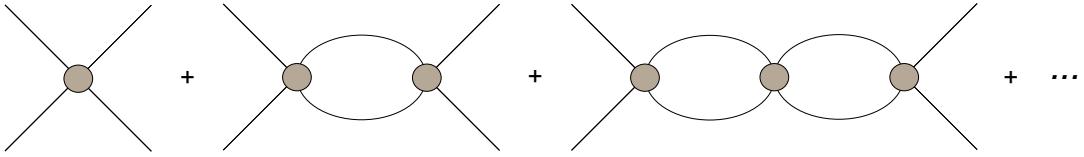


Figure 1: The diagrammatic solution of the LS equation with the effective potential represented by the shaded blob.

First, the physical scattering amplitude that is generated when the potential (2) is iterated is exactly unitary and therefore necessarily contains arbitrarily high powers in energy or momenta. This occurs regardless of the order to which one is working in the momentum expansion of the potential V . This suggests that as long as one is interested in physics near the bound state pole—where exact unitarity is important—the scattering amplitude may be sensitive to physics at arbitrarily short-distance scales. While this observation does not necessarily invalidate the EFT approach, it does mean that one should not simply translate intuition gained about how short-distance physics decouples in perturbative EFT calculations (e.g. in chiral perturbation theory) to the non-perturbative problem at hand. Thus, the first question we face is whether a systematic expansion in the potential translates to a systematic determination of the scattering amplitude. A necessary condition for this to occur is that the integrals in the LS equation, which formally extend up to infinity, be dominated by momenta of order $p \sim k \ll m_\pi$, where k is the on-shell momentum, since otherwise there is no small parameter in the expansion (2) for V .

Second, non-perturbative regularization and renormalization is required when iterating to all orders using the LS equation. This is an issue because of the presence of ultraviolet divergences, which generally can arise in two ways. When explicit pions are included in the EFT, the potential itself may contain loop graphs which require regularization and renormalization. These divergences are easily handled using standard perturbative methods. A second type of ultraviolet divergence arises because in solving the LS equation one integrates the potential over all momenta. Given the hard asymptotic behavior inherent to the momentum expansion this necessarily introduces new divergences. The divergences which arise from iterating the potential become more severe as one goes to higher order in the EFT expansion. These divergences apparently violate the assumption that $p \ll m_\pi$, so the existence of a procedure to regularize these divergences and renormalize in such a way that

the momentum scales probed inside loops are ultimately well below m_π provides a non-trivial condition on the existence of an EFT.

In ordinary perturbative EFT, all regularization schemes lead to the same renormalized amplitude. This insensitivity to the short-distance physics implied by the regulator makes perturbative EFT methods like chiral perturbation theory possible. Similarly, there is no hope of defining a sensible EFT for NN scattering unless there is some degree of regulator independence. An interesting feature that arises when regularizing and renormalizing the Lippmann-Schwinger equation is that not all regulators lead to the same physical results. For instance, we will show that dimensional regularization (DR) and cutoff schemes lead to different physical scattering amplitudes. This result should give practitioners of EFT pause, since it suggests a sensitivity to short-distance physics which violates the basic tenets of EFT. We further demonstrate that in order to generate low-energy (quasi-)bound states in the NN system within this EFT, one must use a scale-dependent regulator.

Cutoff schemes provide the most physical means of regularizing the effective theory. We therefore focus on whether it is possible to implement the regularization and renormalization program in a consistent fashion using cutoff schemes. Within this restricted class of regulators physical results are insensitive to the specific choice of regulator. However, we find that taking the cutoff to infinity requires that the effective range parameter in the scattering amplitude be negative. In physical processes of interest the effective range parameter is positive. The impossibility of maintaining a positive effective range when the cutoff is removed follows from an old theorem of Wigner, which depends only on general physical principles [20, 21]. Therefore, either the cutoff must be kept finite or else all orders in the effective potential must be retained. Using an extension of Wigner's theorem to the EFT where pions are explicitly included, we further argue that the inclusion of pions as explicit degrees of freedom in the EFT does not resolve these difficulties.

Having found that an EFT with only nucleons as explicit degrees of freedom can work only if there is a finite cutoff or regulator scale, we investigate the possibility of a cutoff effective field theory. In cutoff EFT, cutoff dependence in physical observables is removed systematically by adding higher-dimensional operators to the effective action. However, this procedure makes sense only if there is a small parameter which allows one to conclude that higher-order operators in the action are negligible. We show that there is no such small parameter in NN scattering when there are low-energy (quasi-)bound states.

We stress that we *are not* questioning the existence of an effective field theory of NN scattering. Since the principles underlying EFT are causality and locality, an EFT of NN scattering must exist unless a sacred principle is violated. The question we address is: “What are the relevant low-energy degrees of freedom which must be included explicitly in the EFT?” The problem of NN scattering is subtle in this respect because generally there are singularities at unnaturally low energies (e.g., the deuteron) in scattering amplitudes. These singularities are not present as fields in the effective action. What our analysis here shows is that if low-lying bound states are present then there is no systematic EFT for NN scattering in which there are only nucleon fields in the effective Lagrangian. This would suggest the necessity of introducing the physics of (quasi-)bound states in terms of explicit degrees of freedom in the EFT description. A step in this direction has been taken in Ref. [22], but in this paper we will not discuss such ideas further. Therefore, we use the term EFT to

mean an effective field theory in which low-energy bound states are not included as fields in the effective action, i.e. an EFT in which only nucleons (and pions) are explicit degrees of freedom.

In Sec. 2 we discuss the 1S_0 channel in NN scattering and introduce a simple potential model that reproduces data remarkably well. This model demonstrates how an unnaturally small scale can emerge from a theory with only large scales. In Sec. 3 we review the effective field theory power-counting argument for NN scattering, originally proposed by Weinberg. In particular, we examine arguments for power counting in the potential rather than in the amplitude in the NN problem. In Sec. 4 we perform a “leading-order” EFT calculation for NN scattering in the 1S_0 channel and show that all regularization schemes give the same scattering amplitude. In Sec. 5 we perform a “second-order” EFT calculation using cutoff regularization. We show that if a cutoff is introduced such that the regulated theory respects all physical principles, then the cutoff cannot be taken to infinity. We discuss how this result relates to an old theorem of Wigner. In Sec. 6 we compare dimensional regularization (DR) and cutoff regularization schemes. We show that DR and cutoff regularization give different physical scattering amplitudes in the second-order EFT calculation, indicating that there is a sensitivity to the choice of regulator, and therefore to short-distance physics. We find that the presence of a regulator scale is necessary to reproduce unnaturally low-lying (quasi-) bound states. In Sec. 7 we consider the possibility of an effective field theory with a finite cutoff. We find that although a procedure whereby cutoff dependence in the amplitude is removed by adding higher orders in the effective potential can be defined, the small parameter which is necessary to conclude that this is systematic is simply not present. We summarize and conclude in Sec. 8.

2 Phenomenology of the 1S_0 NN Interaction

The 1S_0 phase shift in NN scattering is remarkably well reproduced (see Fig. 2) up to center of mass momenta of order m_π by the first two terms in the effective-range expansion:

$$\frac{1}{T^{\text{on}}(k)} = -\frac{M}{4\pi} \left[-\frac{1}{a} + \frac{1}{2} r_e k^2 - ik \right], \quad (4)$$

where a is the scattering length, r_e is the effective range, and k is the on-shell momentum, $k = \sqrt{ME}$. Experimentally these parameters are determined to be

$$a = -23.714 \pm 0.013 \text{ fm} \quad r_e = 2.73 \pm 0.03 \text{ fm}. \quad (5)$$

The extremely large (negative) value of the scattering length implies that there is a virtual bound state in this channel very near zero energy. While the value of r_e is consistent with what one might expect for a “natural” theory where pions dominate the low-energy physics ($r_e \sim 1/m_\pi$), the value of a is far from being natural ($a \gg 1/m_\pi$).

In order to understand the emergence of an unnaturally large scattering length it is instructive to consider a simple separable potential model whose scattering amplitude exactly reproduces the effective-range expansion, Eq. (4), with no terms of higher order in momenta. Consider the potential

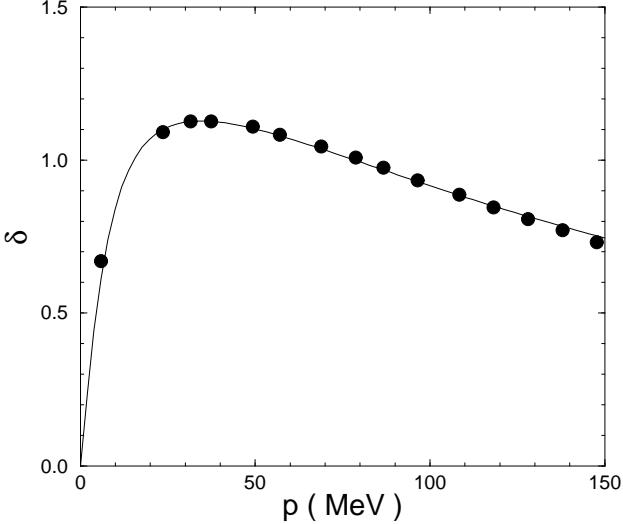


Figure 2: The effective range expansion with the extracted phase-shift data up to center-of-mass momenta of order m_π . Phase shift data are taken from Ref. [23].

$$V(p', p) = -\frac{4\pi g}{\Lambda M} \left(1 + \frac{p'^2}{\Lambda^2}\right)^{-1/2} \left(1 + \frac{p^2}{\Lambda^2}\right)^{-1/2}, \quad (6)$$

where g is a dimensionless coupling constant and Λ is a scale that characterizes the range of the potential. Clearly this model has nothing to do with effective field theory. It is a simple model for the 1S_0 channel which captures some of the salient features of the scattering.

Inserting this potential into the LS equation (3) gives the on-shell amplitude:

$$\begin{aligned} \frac{1}{T^{\text{on}}(k)} &= V(k, k)^{-1} \left[1 - M \int \frac{d^3 q}{(2\pi)^3} \frac{V(q, q)}{k^2 - q^2 + i\epsilon} \right] \\ &= -\frac{M}{4\pi} \left[\frac{\Lambda}{g} \left(1 + \frac{k^2}{\Lambda^2}\right) - (\Lambda + ik) \right]. \end{aligned} \quad (7)$$

This is simply the effective-range expansion to order k^2 . One can easily read off the scattering length and effective range, by comparing Eqs. (7) and (4):

$$a = \frac{1}{\Lambda} \frac{g}{g-1} \quad r_e = \frac{2}{g\Lambda}. \quad (8)$$

The choices $\Lambda = 152$ MeV and $g = 0.95$ then reproduce the values (5).

Note that the effective range is directly proportional to the range, $1/\Lambda$, of the potential. The expression for a makes clear that an unnaturally large scattering length occurs if g is “accidentally” close to one². In this model the large scattering length is the result of a

²See also [22] for an illustrative example.

cancellation between two distinct contributions to the scattering amplitude. The first of these contributions comes from the “strength” of the potential (the blob in Fig. 1) since it originates in the $1/V(k, k)$ term in Eq. (7). The second term comes from virtual excitations in the loops in Fig. 1. This contribution is sensitive to the range of the potential, i.e. $1/\Lambda$. Note that both of these contributions to $1/a$ are natural, i.e. given by $-\Lambda/g$ and Λ respectively. Nevertheless, if $g \approx 1$ there is a cancellation between the “strength” and the “range”, which results in an unnaturally large scattering length. We will see that this qualitative feature persists in the EFT treatment.

3 An Effective Field Theory for NN Scattering

In its standard form chiral perturbation theory is not useful in NN scattering because of the presence of low-lying bound states which imply a breakdown of perturbation theory [12, 13]. Weinberg argued that this breakdown manifests itself through the presence of infrared divergences in reducible loop graphs in the limit of static heavy nucleons. This argument is not entirely correct—it is possible to have a “perturbative” effective field theory for the interaction of heavy particles, provided that there are no low-lying bound or quasi-bound states. (See, for instance Ref. [24].) However, since there *are* low-lying bound states in nuclear systems, Weinberg’s main point remains true: if power-counting arguments are relevant in nuclear physics, they must be applied to an effective potential, rather than to the scattering amplitude. In general, the effective NN potential is the sum of all two-particle-irreducible (2PI) $NN \rightarrow NN$ diagrams. When there are pions and nucleons in the EFT, the effective potential contains pion-exchange diagrams, short-range contact interactions and a mixture of the two. However, in this paper we are interested in energy scales where m_π is large compared to the initial and final momenta p and p' . In this case EFT arguments imply that the pion can be integrated out and replaced by contact interactions.

The starting point of any EFT calculation is an effective Lagrangian, where operators are ordered according to the number of derivatives. The most general effective Lagrangian consistent with spin and isospin, including only operators relevant to 1S_0 scattering is

$$\mathcal{L} = N^\dagger i\partial_t N - N^\dagger \frac{\nabla^2}{2M} N - \frac{1}{2} C(N^\dagger N)^2 - \frac{1}{2} C_2(N^\dagger \nabla^2 N)(N^\dagger N) + h.c. + \dots \quad (9)$$

There are only tree graphs in the NN potential obtained from this Lagrangian. Hence there is a one-to-one correspondence between the operators in the effective Lagrangian and the terms in the potential. When iterated using the Lippmann-Schwinger equation, this potential gives the full NN amplitude of the theory. This procedure is entirely equivalent to solving the quantum partition function with an effective action to a given order in the derivative expansion, and thereby generating the four-point correlation function. There is therefore, at least for this EFT, actually no need to formulate the discussion in terms of potentials. Nevertheless, here we use the familiar language of non-relativistic quantum mechanics.

The effective potential to order ν in the derivative expansion can be expressed in the form [12, 13, 14, 15]

$$V^{(\nu)}(p', p) = \frac{1}{\Lambda^2} \sum_{\bar{\nu}=0}^{\nu} \left[\frac{(p, p')}{\Lambda} \right]^{\bar{\nu}} c_{\bar{\nu}} \quad (10)$$

where the sum here is over all possible terms extracted from (9) and Λ is the scale of the physics integrated out, taken to be m_π in the present context. Equation (10) is intended to be symbolic; (p, p') indicates that either of these quantities may appear in the expansion, in any combination consistent with symmetry, with only their total power constrained. For instance, at $\bar{\nu} = 4$ we have the structures $p^4 + p'^4$ and $p^2 p'^2$ with coefficients c_4 and c'_4 , respectively. It is assumed that the coefficients $c_{\bar{\nu}}$ are natural; that is, of order unity. The fundamental assumption underlying effective field theory for the NN interaction is that this expansion in the potential, or equivalently that in the Lagrangian, may be sensibly truncated at some finite order, ν . It is the purpose of this paper to test this assumption.

The physical scattering amplitude is obtained by iterating the potential (10) using the Lippmann-Schwinger equation

$$T(p', p; E) = V(p', p) + M \int \frac{d^3 q}{(2\pi)^3} V(p', q) \frac{1}{EM - q^2 + i\epsilon} T(q, p; E), \quad (11)$$

which generates the T -matrix. This procedure is illustrated in Fig. 1. By assumption, truncating the expansion (10) and retaining only its first few terms will be valid only for nucleon momenta well below Λ . It is clear that a method of regularizing the otherwise-divergent integrals which occur when potentials such as (10) are inserted into the LS equation must be specified.

4 “Leading order” EFT calculation ($\nu = 0$)

We first investigate these questions by considering the 1S_0 NN amplitude generated by the “zeroth-order” EFT potential. This potential is

$$V^{(0)}(p', p) = C. \quad (12)$$

The solution of the LS equation with this potential is easily found to be

$$\frac{1}{T^{\text{on}}(k)} = \frac{1}{C} - I(k), \quad (13)$$

where $k = \sqrt{ME}$ is the on-shell momentum, and

$$I(k) \equiv M \int \frac{d^3 q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\epsilon}. \quad (14)$$

This integral is linearly divergent. Regularizing with a sharp cutoff, β , gives

$$\text{Re} \left(\frac{1}{T^{\text{on}}(k)} \right) = \frac{1}{C_{\text{cutoff}}} + \frac{M\beta}{2\pi^2}, \quad (15)$$

where we have neglected terms that vanish as powers of $1/\beta$. We choose the renormalization condition

$$\text{Re} \left(\frac{1}{T^{\text{on}}(0)} \right) \equiv \frac{1}{C_{\text{ren}}} = \frac{M}{4\pi a} \quad (16)$$

where C_{ren} is the “renormalized potential” and a is the physical scattering length. Note that just as in the example of Sec. 2 there is a cancellation in Eq. (15) between the contributions to $1/a$ from the strength of the potential, and the integration over loop momenta.

In the limit $\beta \rightarrow \infty$ we then find the physical renormalized scattering amplitude,

$$\frac{1}{T^{\text{on}}(k)} = -\frac{M}{4\pi} \left[-\frac{1}{a} - ik \right], \quad (17)$$

derived by Weinberg [13] and Kaplan *et al.* [16]. Note that this result is completely insensitive to choice of regularization scheme. After renormalization the only piece of the integration over internal loop momenta which survives is the imaginary part, which gives the unitarity branch cut and depends only on the on-shell momentum, k . The real piece of $I(k)$, which comes from integration over virtual nucleon momenta, is absorbed into the constant $1/C_{\text{cutoff}}$. This is exactly how one renormalizes in standard EFT, where power-law divergent pieces of loop integrals are absorbed by bare coefficients in the amplitude. One efficient way to ensure that the power-law divergent parts of loop graphs never enter the calculation is to use dimensional regularization. However, note that if DR is used, as in Ref. [16], then the cancellation seen in Eq. (15) is destroyed, since the linear divergence is zero in DR. Although of no physical consequence here, this effect will prove important at second order in the EFT.

The result (17) appears promising: we have reproduced the scattering length term of the effective-range expansion. Moreover, we see that at “leading order” in the effective field theory calculation the scattering amplitude is regularization-scheme independent; the real part of the integration over loop momenta does not have any effect on the final physical result. In order to test the robustness of this result we must calculate corrections. At the next order the potential has momentum dependence and therefore is more sensitive to short-distance physics.

5 “Next-to-leading Order” EFT calculation($\nu=2$)

From the effective Lagrangian (9) we extract the following “second-order” EFT potential:

$$V^{(2)}(p', p) = C + C_2(p^2 + p'^2). \quad (18)$$

An easy way to solve the LS equation with this potential is to observe that $V^{(2)}$ may be written as a two-term separable potential. The on-shell T-matrix is easily found to be (see Ref. [25] for details):

$$\frac{1}{T^{\text{on}}(k)} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5 + k^2 C_2 (2 - C_2 I_3)} - I(k), \quad (19)$$

where

$$I_n \equiv -M \int \frac{d^3q}{(2\pi)^3} q^{n-3}. \quad (20)$$

The integrals I_3 , I_5 and $\text{Re}(I(k)) = I_1$ are all divergent, and so this scattering amplitude must be regularized and renormalized. It is instructive to carry the renormalization a certain distance without specifying a regularization scheme. We choose as renormalized parameters the experimental values of the scattering length, a , and the effective range, r_e . In other words, we fix C and C_2 by demanding that

$$\frac{1}{T^{\text{on}}(k)} = -\frac{M}{4\pi} \left[-\frac{1}{a} + \frac{1}{2} r_e k^2 + O(k^4) - ik \right]. \quad (21)$$

Unitarity guarantees that the imaginary parts agree. Equating the real parts at $k = 0$ yields

$$\frac{M}{4\pi a} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5} - I_1. \quad (22)$$

Equating the k^2 coefficients gives

$$\frac{M r_e}{8\pi} = \left(\frac{M}{4\pi a} + I_1 \right)^2 \left[\frac{1}{(C_2 I_3 - 1)^2 I_3} - \frac{1}{I_3} \right]. \quad (23)$$

Note that the physical parameters are complicated non-linear functions of divergent integrals and bare parameters. The scattering amplitude can now be written as

$$\text{Re} \left(\frac{1}{T^{\text{on}}(k)} \right) = \frac{M/(4\pi a) - k^2 I_1 A}{1 + k^2 A}, \quad (24)$$

with

$$A \equiv \frac{M r_e}{8\pi} \left(\frac{M}{4\pi a} + I_1 \right)^{-1}. \quad (25)$$

Notice that the scattering amplitude is now expressed in terms of the physical parameters a and r_e and the linearly divergent integral I_1 . It is not surprising that we have a divergence left over after imposing our renormalization conditions since we had three distinct divergences— I_1 , I_3 and I_5 —and only two free parameters— C and C_2 .

5.1 Cutoff Regularization

If we now regularize I_1 using a sharp cutoff, β , we find in the $\beta \rightarrow \infty$ limit:

$$\frac{1}{T^{\text{on}}(k)} = -\frac{M}{4\pi} \left[-\frac{1}{a} + \frac{1}{2} r_e k^2 - ik \right]. \quad (26)$$

which is the effective-range expansion to order k^2 . Given the success of Eq. (26) in describing the 1S_0 NN phase shifts this result seems quite promising. However, there is a subtlety hidden in this result. The second term in Eq. (23) disappears when the cutoff is removed. Consequently, as $\beta \rightarrow \infty$

$$\frac{Mr_e}{8\pi} \rightarrow \frac{1}{I_3} \left(\frac{I_1}{C_2 I_3 - 1} \right)^2, \quad (27)$$

from which we see that $r_e \leq 0$, (recalling that I_n defined in Eq. (20) is negative definite), *no matter what real value we choose for the constant C_2 in the bare Lagrangian*. This result is at first sight rather peculiar since it implies that the sign of r_e is predicted in the effective theory with the cutoff removed. It is the behavior of the power-law divergences arising from integrals over internal loop momenta which leads to the result $r_e \leq 0$. This is in contrast to the “zeroth-order” amplitude, where these integrals over internal loop momenta did not affect the final on-shell amplitude. Moreover, the result $r_e \leq 0$ as $\beta \rightarrow \infty$ clearly means that in order to describe a system with $r_e > 0$ using this potential one cannot remove the cutoff but must instead maintain the hierarchy $\beta \leq O(1/r_e)$. (The necessity of keeping the cutoff small and below $O(1/r_e)$ is supported by direct calculation [17].)

5.2 Wigner’s Bound

The fact that one cannot take the cutoff β to infinity and still obtain a positive effective range is, in fact, not surprising. Long ago, assuming only causality and unitarity, Wigner proved that if a potential vanishes beyond range R then the rate at which phase shifts can change with energy is bounded by [20]

$$\frac{d\delta(k)}{dk} \geq -R + \frac{1}{2k} \sin(2\delta(k) + 2kR). \quad (28)$$

It is straightforward to show that this translates to [21, 26]

$$r_e \leq 2 \left[R - \frac{R^2}{a} + \frac{R^3}{3a^2} \right]. \quad (29)$$

In light of this general result it is no longer surprising that the effective range was found to be negative in the EFT calculation when the cutoff β was removed. Wigner’s bound implies that no theory obeying the general physical principles of causality and unitarity can support a positive effective range with a zero-range force. When using a sharp cutoff the regulated Hamiltonian satisfies all physical principles and is therefore a legitimate physical theory, with the cutoff identified with the range; i.e. $\beta \sim 1/R$. Consequently in the limit $\beta \rightarrow \infty$ one is trying to use zero-range potentials to produce a positive effective range. Equation (29) shows that this is impossible. Therefore, Wigner’s bound indicates that there is no EFT treatment of NN scattering with nucleons as the only explicit degrees of freedom in which it is possible to remove the cutoff from the problem. One might think that including explicit pions in the effective theory description will ameliorate the situation. This is not so; it has been shown that the Wigner bound is a general feature of contact interactions and continues to apply in the presence of pions [17].

6 Dimensional Regularization

As we have seen, *a priori* there are two sources of short-distance physics in the effective field theory calculation: operators in the effective Lagrangian of increasing dimensionality whose coefficients encode information about short-distance physics, and the effects of virtual particle excitations within loop graphs. In ordinary perturbative EFT calculations the existence of a consistent power-counting scheme in the S-matrix relies on removing the short-distance physics that arises in loop graphs via the renormalization procedure [19]. Since the short-distance physics is removed in a manner insensitive to choice of regularization scheme it is economical to use DR to regularize and renormalize, because DR respects chiral and gauge symmetries. When considering the relevance of EFT methods in nuclear physics one might choose to extrapolate intuition gained from perturbation theory and regulate using DR. However, it is important to realize that the use of DR implicitly *assumes* that EFT can work only if the short-distance physics buried in loop graphs *does not* contribute to low-energy physics. Of course there is no need to make this assumption, cutoff regularization is the most physically-transparent method of regularization and keeps track of all sources of short-distance physics in the calculation.

In this section we compare DR and cutoff schemes. There are two fundamental points we wish to make: (i) DR and cutoff regularization do not lead to the same physical scattering amplitude. Therefore in the nonperturbative context of NN scattering, short-distance physics from loops is important, in contrast to the situation in perturbative calculations; (ii) When low-lying (quasi-)bound states are present, a necessary (but not sufficient) condition for a workable EFT treatment of NN scattering is that short-distance effects from loops contribute to the physical scattering amplitude.

We now use dimensional regularization (DR) to regulate the integrals I_1 , I_3 , and I_5 which appear in the expression for T^{on} above. As discussed above, DR is a convenient way to implement the idea that is central to the success of perturbative EFT: the power-law divergent pieces of integrals over internal loop momenta should not affect the final physical scattering amplitude. If this holds true then DR should lead to the amplitude found using cutoff regularization. In DR all power-law divergences vanish, therefore $I_1 = I_3 = I_5 = 0$. Consequently, Eqs. (24) and (25) reduce to

$$\frac{1}{T_{\text{DR}}^{\text{on}}(k)} = -\frac{M}{4\pi} \left[\frac{1}{(-a - \frac{1}{2}r_e a^2 k^2)} - ik \right], \quad (30)$$

in agreement with Kaplan *et al.* [16]. The renormalized scattering amplitude is not the same as that obtained using cutoff regularization. Therefore the amplitude calculated using the second-order EFT potential is not independent of regularization scheme, and so the scattering amplitude must be sensitive to the short-distance physics from loops. This raises the question of whether the EFT treatment based on potentials has really managed to separate the long-distance physics from the short-distance physics. This separation is a precondition for the successful application of EFT ideas.

Clearly the DR amplitude maps to the effective-range expansion only for momenta $k \ll 1/\sqrt{ar_e}$. This is consistent with Eq. (26) if both a and r_e are natural, as then Eqs. (30) and (26) will agree within the domain of validity of the EFT, $k^2 \ll \Lambda^2$. The effective theory is then

perturbative. However, if the scattering length is unnaturally large the momentum domain within which the two forms are equivalent becomes small. In fact, Eq. (30) reproduces the data in the 1S_0 extremely poorly, since it only agrees with the phenomenologically efficacious amplitude (4) for very small k . It was argued elsewhere that this failure implies that the effective theory calculation is not valid in the presence of a large scattering length [16].

However, this failure of dimensional regularization is no surprise when one considers the simple model of the effective-range expansion discussed in Sec. 2. There it was clear that an unnaturally large scattering length occurs through the cancellation between two terms in the scattering amplitude, one of which arises from integration over loop momenta. This cancellation between “range” and “strength” is a general feature whenever one solves the Schrödinger equation with some finite-range potential. By its very nature, DR *cannot* give a scattering amplitude that feels this cancellation, since it discards all short-distance physics that comes from loops. In fact, since there are no logarithmic divergences in this problem and the loop graphs have no finite real part, it is straightforward to show that DR gives an amplitude in which only the absorptive parts of the loop graphs are retained³. (See Fig. 3.) Hence when there are low-lying bound states the bound-state energy sets the scale of the coefficients in the potential of the dimensional regularization calculation. But recall that this was precisely the problem that iterating the potential via the LS equation was supposed to avoid. Thus the implementation of the general cancellation between range and strength must be an element of any EFT description of NN scattering.

In general, information about the range of the potential enters through these power-law divergences, and DR’s neglect of all power-law divergences means it discards this information on the range of the interaction. This is no accident, DR is designed to be a scale-independent regularization scheme. This aspect of the DR prescription allows it to violate the Wigner bound and give an amplitude which has a positive effective range from an EFT potential which is apparently zero range.

Therefore we argue that a regulator that introduces a new scale in NN scattering must be used. However, in the next section we will show that effects due to this new scale actually destroy power counting. Therefore, we believe that the intuition that EFT can work only if scattering amplitudes are insensitive to the short-distance physics arising from loop graphs, which is the primary motivation for using DR, is actually correct. In other words, in all circumstances in which EFT is systematic DR can, and for simplicity should, be used. In the calculation of NN scattering in the presence of low-energy bound states DR fails only because the sensitivity to short-distance physics inherent in this problem means that no sensible EFT can be formulated for this problem.

7 Power Counting in Cutoff Effective Field Theory

The arguments of the previous section show that an EFT approach to NN scattering will only be efficacious in the presence of low-lying (quasi-)bound states if the short-distance physics in loop graphs contributes to the physical scattering amplitude. If a cutoff is introduced, so that we can keep track of this physics, the bound (29) indicates that it cannot be taken to

³This point has also recently been made by Lepage [18].

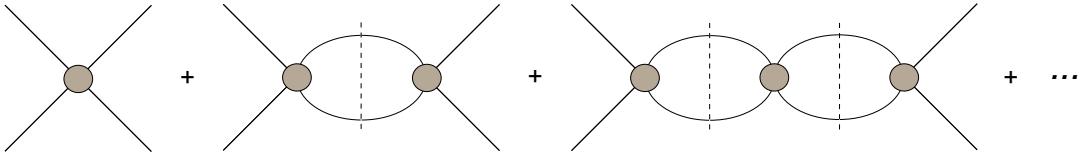


Figure 3: The diagrammatic solution of the dimensionally regulated LS equation with potential represented by the shaded blob. Only the absorptive parts of the loop graphs remain.

infinity. Therefore, in this section we investigate the possibility of an EFT for NN scattering in the presence of a finite cutoff.

This physically intuitive approach has been advocated by Lepage [18, 27]. The idea is that one takes an underlying theory of NN interactions and introduces a (sharp or smooth) momentum cutoff β representing the scale at which the first new physics becomes important. All loops now only include momenta $p < \beta$. Of course, one must compensate for the effects of these neglected modes. However, Lepage argues that since these modes are highly virtual, one may approximate their effects by a sequence of local contact interactions. Furthermore, if the cutoff β is placed well below the mass Λ of some exchanged quantum, then, for momenta \mathbf{p} and \mathbf{p}' below the cutoff, the exchange of this quantum:

$$V_\Lambda(\mathbf{p}', \mathbf{p}) \sim \frac{1}{(\mathbf{p}' - \mathbf{p})^2 + \Lambda^2}; \quad (31)$$

may be replaced by a contact interaction, since $p', p < \beta \ll \Lambda$. Therefore the effects coming from exchanges of quanta with masses well above the cutoff scale β may also be approximated by contact interactions. For the numerical application of these ideas to the NN problem see [15, 17, 18].

Now, all that has been said in the previous paragraph still applies if we set our cutoff β below the scale m_π . Then the only explicit degrees of freedom in the problem are nucleon modes with momentum $\beta < m_\pi$. All higher-momentum nucleon modes *and* all exchanged mesons are integrated out. This cutoff effective field theory of the NN interaction is of little practical use, but can be investigated analytically in a way that raises issues of principle. The effective Lagrangian is

$$\mathcal{L} = N^\dagger i\partial_t N - N^\dagger \frac{\nabla^2}{2M} N - \frac{1}{2} C(\beta) (N^\dagger N)^2 - \frac{1}{2} C_2(\beta) (N^\dagger \nabla^2 N) (N^\dagger N) + h.c. + \dots \quad (32)$$

where the dots signify operators with more than two derivatives. As discussed in Sec. 3, in the limit that the nucleon is very heavy we can formally calculate the four-point function exactly. To do this we merely write down the effective potential which corresponds to this Lagrangian:

$$V(p', p) = [C + C_2(p^2 + p'^2) + \dots] \theta(\beta - p) \theta(\beta - p'), \quad (33)$$

and iterate via the Lippmann-Schwinger equation (11). Here the theta function imposes a sharp cutoff, and so all integrals (not just the divergent ones) will be cut off sharply at momentum β . After renormalization the coefficients C , C_2 , etc., will, of course, depend on the cutoff scale β , as well as physical scales in the problem.

Of course, the expression (33) is an infinite series, and for practical computation some method of truncating it must be found. The fundamental philosophy of cutoff EFT provides a rationale for this as follows. If we work to any finite order in the effective potential, cutoff-dependent terms in the scattering amplitude will appear. These must be in correspondence with neglected higher-order operators in V . If such terms are progressively added to V , one may remove the cutoff dependence order-by-order. Below we will see that one can indeed define such a “systematic” procedure in this problem. However, *in order that it really make sense to truncate the effective potential (33) at some finite order it must be that the operators that are neglected are in some sense small.*

Naively one would expect that a sufficient condition for the operator with n derivatives to be a small correction to the overall potential is:

$$\hat{p}^{2(n-m)} \ll \frac{C_{2m}}{C_{2n}}; \quad \forall m \geq 0 \text{ such that } n > m, \quad (34)$$

where \hat{p} is the momentum operator. This operator yields some characteristic momentum in the problem. Now, in perturbation theory the short-distance physics can be removed from loops via renormalization, thereby achieving a clean separation of scales, and so the characteristic momentum is simply the external momentum. Thus, in perturbative calculations the operator hierarchy (34) can be maintained, as long as one restricts oneself to small enough external momenta.

However, in a nonperturbative context we have already seen that physics from loops must enter into the calculation if low-lying (quasi-)bound states are to be generated. Consequently, we expect that momenta at scales $\hat{p}^2 \sim \beta^2$ are important. Below we show that

$$\frac{C_2}{C} \sim \frac{1}{\beta^2}. \quad (35)$$

More generally, condition (35) becomes:

$$\frac{C_{2m}}{C_{2n}} \sim \beta^{2(n-m)}. \quad (36)$$

Combining Eqs. (36) and (34) we obtain

$$(\hat{p}/\beta)^j \ll 1; \quad j = 2, 4, 6, \dots \quad (37)$$

Equation (37) reflects the fact that the crucial momenta in the problem must be well below the scale where the cutoff effective field theory breaks down. The external momentum can certainly be kept in this energy regime. However, below we also show that, even if the external momentum scales are small, the quantum mechanical averages of operators $\langle \hat{p}^j \rangle$ involve β . In this circumstance there is little justification for truncating the effective Lagrangian and keeping only a finite number of operators.

7.1 “Leading order” cutoff EFT calculation ($\nu = 0$)

Consider once again the 1S_0 NN amplitude generated by the “zeroth-order” EFT potential. This potential in cutoff EFT is given by

$$V^{(0)}(p', p) = C^{(0)}\theta(\beta - p)\theta(\beta - p') \quad (38)$$

The solution of the Lippmann-Schwinger equation is then

$$\frac{1}{T^{\text{on}}(k)} = \frac{1}{C^{(0)}} - I_\beta(k), \quad (39)$$

where

$$I_\beta(k) = -M \int \frac{d^3q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\epsilon} \theta(\beta - q). \quad (40)$$

Matching the solution to the effective-range expansion gives

$$\frac{1}{T^{\text{on}}(k)} = -\frac{M}{4\pi} \left[-\frac{1}{a} + \frac{1}{2} r_e^{(0)} k^2 + O(k^4) - ik \right], \quad (41)$$

where the renormalization condition for $C^{(0)}$ is still Eq. (15). Note that a is cutoff independent as before, but now we have a non-zero effective range in this zeroth-order calculation:

$$r_e^{(0)} = \frac{4}{\pi\beta}. \quad (42)$$

This $r_e^{(0)}$ arises from the energy dependence introduced by the cutoff. The appearance of a cutoff-dependent quantity like $r_e^{(0)}$ in the renormalized amplitude only reflects the limited accuracy of this calculation. Below we will see that r_e can be made cutoff independent by including higher-order terms in the effective potential.

7.2 “Next-to-leading Order” cutoff EFT calculation ($\nu=2$)

We now turn our attention to the amplitude and renormalization conditions that arise when one takes the “second-order” effective potential:

$$V^{(2)}(p', p) = [C + C_2(p^2 + p'^2)]\theta(\beta - p)\theta(\beta - p'), \quad (43)$$

and iterates it via the Lippmann-Schwinger equation. Repeating the analysis of Sec. 5, but now with all integrals cutoff sharply at momentum β , we again obtain the amplitude:

$$\frac{1}{T^{\text{on}}(k)} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5 + k^2 C_2 (2 - C_2 I_3)} - I_\beta(k), \quad (44)$$

while I_3 , and I_5 are as before, provided that it is understood that a sharp cutoff is to be used to render the divergences finite. From these equations we obtain the following equations

for C and C_2 :

$$\frac{M}{4\pi a} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5} - I_1; \quad (45)$$

$$\frac{Mr_e}{8\pi} = \left(\frac{M}{4\pi a} + I_1 \right)^2 \frac{C_2(2 - C_2 I_3)}{(C_2 I_3 - 1)^2} + \frac{Mr_e^{(0)}}{8\pi}, \quad (46)$$

where, as above, the last term in Eq. (46) arises because the presence of the cutoff generates additional energy dependence when the integral $I_\beta(k)$ is evaluated.

Of course, as β is varied the C and C_2 that satisfy Eqs. (45) and (46) will change significantly. However, because one is fitting to low-energy scattering data different values of β will not lead to any fundamental differences in the low-energy T-matrix. Since C and C_2 are fit to the first two terms in the effective-range expansion sensitivity to the cutoff enters first through the fourth-order term in the effective-range expansion. Therefore the effects of choosing different cutoffs can only appear in the on-shell amplitude at order $(k/\beta)^4$, where k is the on-shell momentum.

This is precisely the sense in which working to higher order in the potential will improve the fit “systematically”. If we constructed the “next-order” effective potential, $V^{(4)}$ and refitted the (four) coefficients appearing in it to the low-energy data then sensitivity to the cutoff will be pushed back to an $O((k/\beta)^8)$ correction. Thus, the introduction of higher-dimension operators allows for the “systematic” removal of cutoff dependence in the amplitude. In general, the two-body scattering amplitude will be insensitive to the choice of regulator, provided one restricts oneself to the domain of validity of the cutoff effective field theory, $k \ll \beta$.

This implies that so long as we renormalize the coefficients to the low-energy scattering data we “systematically” improve our fits to the scattering data. This is the systematicity seen in the work of Lepage [18], and that of Scaldeferri *et al.* [17]. Nevertheless, this does not yield a systematic EFT potential in the sense we have defined it here. After all, *any* parameterization of a potential which is rich enough to fit the lowest terms in the effective-range expansion will similarly be improved as one adds parameters to it. The question is whether one can use the power counting to argue *a priori* that certain contributions to the potential of Eq. (33) will be systematically small and hence can be neglected at some specified level of accuracy.

To answer this question we must examine the values of C and C_2 which are required to solve the equations above. Assuming that we are working in the regime where $1/a \ll \beta$ it follows that the second of these two equations becomes

$$\frac{M}{8\pi}(r_e - r_e^{(0)}) \approx \frac{I_1^2}{I_3} \frac{\overline{C}_2(2 - \overline{C}_2)}{(\overline{C}_2 - 1)^2}, \quad (47)$$

where $\overline{C}_2 \equiv C_2 I_3$. This leads to a quadratic equation for \overline{C}_2 , which for values of β up to some β_{\max} has real solutions. For $\beta > \beta_{\max}$ the renormalization condition for C_2 has no real solution if $r_e > 0$, as discussed in Sec. 5 and Ref. [25]. However, if $\beta \ll 4/(\pi r_e)$, then we see that $\overline{C}_2 = 1 \pm \frac{\sqrt{3}}{2}$. Consequently, we infer

$$C_2 \sim \frac{1}{M\beta^3} \Rightarrow C \sim \frac{1}{M\beta}. \quad (48)$$

This behavior for C is the same as that obtained in the zeroth-order calculation in the case $1/a \ll \beta$. Since we demand that the physical observable a be the same in both the zeroth and second-order calculations, the renormalization conditions (15) for $C^{(0)}$ and (45) for C give

$$\overline{C}^{(0)} = \frac{1}{(1 - \overline{C}_2)^2} \left(\overline{C} + \frac{9}{5} \overline{C}_2^2 \right), \quad (49)$$

where \overline{C} ($\overline{C}^{(0)}$) is I_1 times the original dimensionful coefficient. Now $\overline{C}_2 \equiv C_2 I_3 = 1 \pm \frac{\sqrt{3}}{2}$, provided that $\beta \ll 4/(\pi r_e)$. Thus, Eq. (49) shows that if $1/a \ll \beta \ll 4/(\pi r_e)$, \overline{C} differs significantly from the lowest-order result. Hence, if both the zeroth and second-order effective potentials are to reproduce the scattering length then there must be a large difference between them. This already suggests that a systematic truncation of the cutoff EFT may not be possible.

Note that such problems do not arise if a is natural, i.e. of order $1/m_\pi$, and β is chosen to be much less than m_π ; then the leading order behavior of the coefficients C and C_2 is very different. In fact,

$$C \sim \frac{1}{Mm_\pi}; \quad C_2 \sim \frac{1}{Mm_\pi^2 \beta}. \quad (50)$$

In this case all loop effects coming from C_2 are suppressed by a factor of at least $(\beta/m_\pi)^2$. Indeed, all loop effects are suppressed by a factor of at least β/m_π . Therefore, if $\beta \ll m_\pi$ a non-perturbative calculation is not necessary. In other words, if the experimental parameters are natural then cutoff field theory with $\beta \ll m_\pi$ gives a perturbative EFT in which loop graphs are consistently suppressed [10]. However, if a perturbative calculation is performed then the regularization scheme chosen becomes immaterial, as the short-distance physics may be renormalized away. Dimensional regularization would be an equally valid, and considerably simpler, way of implementing a systematic perturbative EFT. Therefore we do not consider the case where $1/a$ is of order m_π any further here, but instead return to the case where $1/a$ is unnaturally small, as is true in the 1S_0 and $^3S_1 - ^3D_1$ channels in nature.

For unnaturally long scattering lengths Eq. (48) shows that:

$$\frac{C_2}{C} \sim \frac{1}{\beta^2}. \quad (51)$$

Consequently the condition (34) becomes, for $n = 2$, $\hat{p}^2 \ll \beta^2$. It is easy to give a heuristic justification of why this behavior of the coefficients arises in a non-perturbative cutoff EFT calculation, and why we expect similar behavior to all orders in the effective potential. After all, the choice of a theta function to regulate the momentum-space integrals as in Eq. (33) is entirely arbitrary. All that has been said above could be reformulated with a smooth cutoff. This would result in an effective potential of the form

$$V(p', p) = [\tilde{C} + \tilde{C}_2(p^2 + p'^2) + \dots]g(p^2/\beta^2, p'^2/\beta^2) \quad (52)$$

where $g(x, y)$ obeys $g(0, 0) = 1$, $g(x, y) = g(y, x)$ and $g(x, y) \rightarrow 0$ faster than any power of x as $x \rightarrow \infty$ with y held fixed. In a non-perturbative calculation the effective potential should be essentially unaltered by this change in the form of the cutoff. However, this necessarily

means that the ratios \tilde{C}_{2n}/\tilde{C} differ from those C_{2n}/C by terms of order $1/\beta^{2n}$. Therefore for a generic cutoff function g the ratio C_{2n}/C must be of order $1/\beta^{2n}$.

Now, if the ratio C_{2n}/C goes like $1/\beta^{2n}$, then the condition (34) becomes Eq. (37). However, the effective potential is to be used in a momentum regime which extends up to β , and at the upper end of this momentum regime it is clear that all terms in the expansion for V are equally important.

Of course, if internal loops were dominated by the external momentum, k , and so $\hat{p} \approx k$, then this behavior of the coefficients would not be cause for concern, since $k \ll \beta$ could be maintained. However, virtual momenta up to β flow through all internal loops, and we have already argued that the final amplitude is sensitive to these virtual effects which come from the range of the underlying interaction. Therefore we believe it makes sense to consider a quantum average in evaluating the condition (34), since such an average is sensitive to these virtual effects.

Although there is no bound state in this channel, all arguments about the size of operators in the effective action would apply equally well if there was a low-energy bound state in the channel under consideration. So, let us evaluate quantum averages of the operator \hat{p}^{2n} using the bound-state wave function obtained from the zeroth-order potential given by Eq. (38). We use a bound-state wave function because scattering wave functions are not easily normalizable, and so quantum averages become difficult to calculate. The zeroth-order potential yields a wave function for the bound state of energy $E = -B$,

$$\psi^{(0)}(p) = \mathcal{N} \frac{M}{MB + p^2} \theta(\beta - p), \quad (53)$$

where \mathcal{N} is some normalization constant that is yet to be determined. In the regime $MB \ll \beta^2$ this wave function gives

$$\frac{\langle \hat{p}^{2n} \rangle}{\beta^{2n}} \equiv \frac{\langle \psi^{(0)} | \hat{p}^{2n} | \psi^{(0)} \rangle}{\beta^{2n}} = \frac{4}{(2n-1)\pi} \frac{\sqrt{MB}}{\beta}, \quad n = 1, 2, \dots \quad (54)$$

Thus, condition (37) is apparently satisfied. However, if this condition is to hold as an operator equation it must be that, for any $n > 0$:

$$\frac{1}{\beta^{2(n-m)}} \frac{\langle \psi^{(0)} | \hat{p}^{2n} | \psi^{(0)} \rangle}{\langle \psi^{(0)} | \hat{p}^{2m} | \psi^{(0)} \rangle} \ll 1 \quad \forall m \geq 0 \text{ such that } n > m. \quad (55)$$

For $m > 0$ this does not hold. That is to say, if we calculate $\langle V \rangle$ with the wave function $\psi^{(0)}$ there is no reason to truncate the expansion at any finite order, since all terms beyond zeroth order contribute with equal strength to the quantum average.

If there was systematic power counting for the NN potential in cutoff field theory then the contribution of these “higher-order” terms in the potential should get systematically smaller as the “order” is increased. However, it is clear that this does not happen—rather, all terms beyond zeroth order contribute to the potential at the same order. Therefore one cannot justify a truncation of Eqs. (33) and (52) at some finite order in p and p' . For the reasons explained above, such a truncation may result in a good fit to the experimental data for on-shell momenta $k \ll \beta$, but it is not based on a systematic expansion of the NN potential in powers of momentum.

8 Discussion

Our results are discouraging for any effective field theory description of NN scattering in which low-lying bound states are not included as explicit degrees of freedom. These low-energy bound and quasi-bound states exist in nuclear physics, and so, if EFTs in which they are not explicitly included are to be useful, power-counting arguments can only apply to the NN potential, rather than to the NN amplitude. Solving such an EFT with nucleons would appear to be a simple problem in non-relativistic quantum mechanics. However, the problem is complicated by the appearance of power-law divergences which arise from the internal loop integrations in the Lippmann-Schwinger equation. In perturbative EFTs such divergences are renormalized away and have no effect on the physical amplitude. In fact, if this were not the case there would be no consistent power counting in these theories. By contrast, when the second-order EFT potential is iterated to all orders, we find that the resulting scattering amplitude is inevitably regularization scheme dependent: dimensional regularization and cutoff regularization give different renormalized scattering amplitudes. We are therefore forced to conclude that the physical scattering amplitude is sensitive to the power-law divergences arising from loop integrations. One way to understand this sensitivity is to realize that when cutoff regularization is used the potential acquires a range, which is naively $1/\beta$. When the Schrödinger equation is solved all physical observables depend on the upper limit in internal loop integrations, β . In any perturbative calculation this dependence can be removed by the introduction of a finite number of higher-dimensional operators in the Lagrangian. However, for some observables, e.g. the effective range, this cannot be achieved for arbitrary cutoff in our non-perturbative calculation: these observables still depend on the cutoff (equivalently, on the “range” of the potential), even after renormalization, as evidenced by the bound (29). When dimensional regularization discards all power-law divergences in order to maintain its scale independence it removes, by fiat, the power-law divergences which represent an important part of the physics of the range of the NN interaction. This suggests DR should not be used to regularize this potential.

It is straightforward to show that if DR is used to regularize a potential which generates low-energy bound or quasi-bound states, the resulting coefficients in the dimensionally regularized potential will be governed by the bound-state energy, and not by the underlying scale of the EFT [24]. Thus, if DR is used to regularize the EFT NN potential the radius of convergence of the resulting expansion will be small—as seen in the work of Kaplan *et al.* [16]. Again, this may be thought of as a consequence of DR’s discarding power-law divergences: such power-law divergences implement the cancellation between “range” and “strength” which was seen in the simple model of Sec. 2 (see also Ref. [22]) to be the key to allowing a theory with natural scales to produce “unnaturally” low-lying bound states. *Therefore the only hope for a systematic and successful EFT description of NN scattering with nucleons alone lies with methods of regularization which preserve information on these power-law divergences.* Cutoff regularization is such a regularization scheme. However, the effective range is positive in nature, and so we discover that if we wish to use an effective field theory with a cutoff to describe the real world we must keep the cutoff finite and below the scale Λ of the short-distance physics. While this may provide a procedure for fitting low-energy scattering data, such an approach can only be regarded as systematic if the im-

portant momenta inside internal loops are well below the cutoff scale. If this is not true physical observables will be sensitive to details of the potential in precisely the region where the artificially-introduced cutoff plays an important role. As we saw in Sec. 7 there will then be no justification for keeping some terms in the EFT potential and neglecting others. Now, while the external momentum k can clearly be kept small compared to the cutoff, we have just argued that in order to have a successful EFT in the presence of low-lying bound states the virtual momenta of order the cutoff must play a significant role! This becomes apparent mathematically through the strong cutoff dependence of the quantum averages of momentum operators. *We therefore conclude that there is no small parameter which allows one to consistently truncate the regularized NN potential at some finite order in the derivative expansion.*

The preceding argument that power counting fails might be regarded as incomplete since we have demonstrated a violation of power counting at the level of the bare parameters in the regularized potential. In general one does not expect power counting in effective field theories to be respected at the level of the bare parameters in the Lagrangian but rather at the level of the renormalized parameters. For example, in ordinary chiral perturbation theory, the bare parameters which enter the perturbative calculations are strictly infinite, while the renormalized amplitudes have natural power counting. Here we have shown that sensible regularization schemes inevitably involve a finite cutoff and therefore the only object in which systematic power counting can be defined is the potential with bare parameters; *there simply is no renormalized potential*. One might attempt to *define* the renormalized potential as the K matrix, $(\text{Re}[T^{-1}])^{-1}$. This quantity is certainly well defined. However, as discussed elsewhere in this paper, the entire reason for doing EFT at the level of the potential is that we are interested in developing an EFT in the vicinity of the low-energy pole, and power counting in T or K fails in the presence of this pole.

It might also be suggested that the inclusion of pions as explicit degrees of freedom in the effective field theory will ameliorate the situation. However, we do not believe this to be the case. After all, provided the energy under consideration is low enough, nothing in the general EFT arguments applied to the NN potential in Refs. [12, 13, 16] requires the inclusion of explicit pions. When pions are added as fields in the Lagrangian of the EFT the basic object of the calculation remains an EFT potential which is to be iterated via the Lippmann-Schwinger equation. This potential still requires regularization. If cutoff regularization is used a generalization of the Wigner bound implies that even in this EFT with explicit pions the cutoff cannot be taken to infinity [17, 21]. DR could also be used to render finite the divergent integrals which arise upon iteration, but again, doing this implicitly assumes that details of the short-distance physics do not affect the physical scattering amplitude. (In fact, additional problems, beyond those discussed here, arise when DR is applied to the EFT with pions, as noted in Ref. [16].) Therefore it seems that cutoff EFT, with a finite cutoff β , is the safest way to regularize the EFT with explicit pions. Whether the inclusion of explicit pions modifies our conclusions about cutoff EFT is a matter for numerical investigation. Previous numerical calculations indicate that the inclusion of explicit pions does not change the scales of the coefficients C and C_2 [17]. Hence, it does not appear that the explicit inclusion of pions in the EFT will modify our conclusion that there is no systematic EFT for NN scattering.

Richardson *et al.* have independently argued that the EFT approach to NN scattering is not systematic [28]. However, their definition of systematic is more restrictive than ours. They define systematicity to mean that the scattering length a should receive no contribution from the term with two derivatives in the effective Lagrangian (the coefficient C_2). Since a does receive a contribution from this term (see Eq. (45)) they claim that the EFT (9) is not systematic. This definition of systematicity is patterned on perturbative EFT, and may be unduly rigid. We would argue that the appearance of C_2 in a does not of itself imply non-systematicity. After all, the contribution of C_2 to a might be systematically small. This happens, for instance, if a finite cutoff $\beta \ll m_\pi$ is used to regularize the divergences and the physical observables are all natural. On the other hand, in the case of interest here, where $1/a$ is unnaturally large, we saw in the previous section that the contribution of the appropriately-rescaled C_2 to a is of order one. The coefficients C that are obtained in the zeroth and second-order calculations then differ significantly. Perhaps this is only symptomatic of the more general malaise already diagnosed: the higher-order terms in the effective action cannot be safely ignored. In summary, the appearance of higher-order coefficients in the expression for low-momentum observables does not necessarily imply non-systematicity. But here we expect the contribution of these higher-order operators to low-momentum observables to be large, because the effective potential cannot be systematically truncated.

The potential of EFT for the NN interaction lies partly in embedding the NN scattering amplitude derived in the EFT in scattering processes involving three or more nucleons. One would hope to apply power-counting arguments to determine those contributions that are important in the nuclear force and those that are systematically suppressed. In principle, this approach has predictive power. For example, as noted by Weinberg, in a nucleus three-body forces are characteristically down by two powers of p/Λ compared to two-body forces, and so might be expected to yield small contributions to nuclear observables. Furthermore, four-body forces are down by four powers of p/Λ and thus could be expected to be negligible [12, 13]. This suggests that at some reasonably crude level one might simply neglect N -body force effects, $N \geq 3$ in nuclei. An analysis of realistic phenomenological nuclear force calculations [14] and the relative contributions of N -body forces to nuclear binding energies shows qualitative agreement with this prediction. Unfortunately, in light of the results of this paper one must question in what sense it is valid to use power counting to deduce a hierarchy of N -body forces. If EFT power counting is a valid way to deduce that certain terms in the Hamiltonian are suppressed then we would have seen that higher-derivative terms in the potential were negligible. In fact, in none of the regularization schemes discussed here was this the case.

Nevertheless, we stress that effective field theory continues to be the most promising method of systematizing nuclear physics. What we have found here is that an EFT for NN scattering with only nucleons must contain *all* operators in the effective action, and therefore is not useful. This has a simple physical interpretation. The NN scattering amplitude is sensitive to all operators in the effective action because it has a singularity corresponding to the (quasi-)bound state pole. By definition, this pole “feels” all distance scales and therefore naturally requires that operators to all orders in momentum in the effective action be present in order to describe it. The only way around this dilemma is to include the (quasi-)bound state pole as a “fundamental” degree of freedom in the effective

theory, while retaining the four-point NN interactions of the Lagrangian (9) [22]. In the resulting EFT, these NN contact interactions can be treated perturbatively with consistent power counting. Application of these ideas in the three-nucleon scattering problem is being pursued in Ref. [29].

Acknowledgments

We thank M. K. Banerjee, M. C. Birse, W. D. Linch, M. A. Luty, J. A. McGovern, E. F. Redish, U. van Kolck, S. J. Wallace, and S. Weinberg for useful and enjoyable discussions on the topics discussed in this paper. This research was supported in part by the U. S. Department of Energy, Nuclear Physics Division (grant DE-FG02-93ER-40762).

References

- [1] M. M. Nagels, T. A. Rijken, and J. J. de Swart, Phys. Rev. D **17**, 768 (1978).
- [2] M. Lacombe *et al.*, Phys. Rev. C **21**, 861 (1981).
- [3] R. Machleidt, K. Holinde, and C. Elster, Phys. Rep. **149**, 1 (1987).
- [4] R. Machleidt, Adv. Nucl. Phys. **19**, 189 (1989).
- [5] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. C **49**, 2950 (1994), [nucl-th/9406039](#).
- [6] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C **51**, 38 (1995), [nucl-th/9408016](#).
- [7] R. Machleidt, F. Sammarucca, and Y. Song, Phys. Rev. C **53**, 1483 (1996), [nucl-th/9510023](#).
- [8] V. G. J. Stoks and T. A. Rijken, Nucl. Phys. **A613**, 311 (1997), [nucl-th/9611002](#).
- [9] D. B. Kaplan, *Effective Field Theories*, [nucl-th/9506035](#).
- [10] A. Manohar, *Effective Field Theories*, [hep-ph/9506222](#).
- [11] V. Bernard, N. Kaiser, and U.-G. Meißner, Int. Jour. of Mod. Phys. E **4**, 193 (1995), [hep-ph/9501384](#).
- [12] S. Weinberg, Phys. Lett. B. **251**, 288 (1990).
- [13] S. Weinberg, Nucl. Phys. **B363**, 3 (1991).
- [14] J. L. Friar, Few Body Syst. **99**, 1 (1996), [nucl-th/9601012](#).
- [15] C. Ordóñez, L. Ray, and U. van Kolck, Phys. Rev. C **53**, 2086 (1996), [nucl-th/9511380](#).

- [16] D. B. Kaplan, M. Savage, and M. B. Wise, Nucl. Phys. **B478**, 629 (1996), [nucl-th/9605002](#).
- [17] K. A. Scaldeferri, D. R. Phillips, C.-W. Kao, and T. D. Cohen, Phys. Rev. C **56**, 1 (1997), [nucl-th/9610049](#).
- [18] G. P. Lepage, *How to renormalize the Schrödinger equation*, [nucl-th/9706029](#).
- [19] S. Weinberg, Physica A **96**, 327 (1979).
- [20] E. P. Wigner, Phys. Rev. **98**, 145 (1955).
- [21] D. R. Phillips and T. D. Cohen, Phys. Lett. B. **390**, 7 (1997), [nucl-th/9607048](#).
- [22] D. B. Kaplan, Nucl. Phys. **B494**, 471 (1997), [nucl-th/9610052](#).
- [23] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. C **48**, 792 (1993).
- [24] M. Luke and A. V. Manohar, Phys. Rev. D **55**, 4129 (1997), [hep-ph/9610534](#).
- [25] D. R. Phillips, S. R. Beane, and T. D. Cohen, [hep-th/9706070](#).
- [26] C. J. Fewster, J.Phys.A **28**, 1107 (1995), [hep-th/9412050](#).
- [27] G. P. Lepage, in *From Actions to Answers* (World Scientific, Singapore, 1990).
- [28] K. G. Richardson, M. C. Birse, and J. A. McGovern, [hep-ph/9708435](#).
- [29] P. Bedaque, D. B. Kaplan, and U. van Kolck, in preparation.